Note

Generalized Pöschl–Teller potential

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From the point of view of the theory of differential equations, we present a four-parameter exactly solvable generalized Pöschl–Teller potential, related to the Jacobi polynomials, using the previously unconsidered equations.

A simple method for obtaining the solutions to the Schrödinger equation was proposed by Bhattacharjie and Sudarshan [1]. Recently, Lévai [2] has used this idea for generating exactly solvable problems in non-relativistic quantum mechanics, and a number of solvable potentials have been reported [1–4]. The purpose of this note is to add a new type of solvable potentials to the already existing ones.

The Schrödinger equation is

$$\frac{\mathrm{d}^2\Psi(r)}{\mathrm{d}r^2}+[E-V(r)]\Psi(r)=0\,, \tag{1}$$

where

$$E=\frac{2\mu}{\hbar^2}\epsilon \quad \text{and} \quad V(r)=\frac{2\mu}{\hbar^2}v(r)\,.$$

Bhattacharjie and Sudarshan [1] considered the solution of the Schrödinger equation to be

$$\Psi(r)=f(r)F(g(r))\,, \tag{2}$$

where $F(g(r))$ is a function which satisfies the second-order differential equation

$$\frac{\mathrm{d}^2F(g)}{\mathrm{d}g^2}+Q(g)\frac{\mathrm{d}F(g)}{\mathrm{d}g}+R(g)F(g)=0\,. \tag{3}$$

Choosing $Q(g(r))$ and $R(g(r))$, eq. (3) is reduced to a special case of the hypergeometric equation [5]. Substituting (2) into (1) and comparing with (3) leads to that $f(r)$ is given by

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$$f(r) = N(g')^{-1/2} \exp\left[\frac{1}{2}\int^{g} Q(g)\,\mathrm{d}g\right]. \tag{4}$$

After eliminating $f(r)$, expressed explicitly in form (4), one can easily construct $E - V(r)$ by comparing (1) and (3) in terms of $g(r)$, $Q(g(r))$ and $R(g(r))$:

$$E - V(r) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + (g')^2\left[R(g) - \frac{1}{2}\frac{\mathrm{d}Q(g)}{\mathrm{d}g} - \frac{1}{4}Q^2(g)\right]. \tag{5}$$

The idea is to find the fraction of the right-hand side of (5) corresponding to the potential and energy. The forms of $Q(g(r))$ and $R(g(r))$ are well defined for any solution $F(g(r))$ of a hypergeometric equation [5]. It transpired [2–4] that a number of solvable potentials can be obtained by letting the solutions to eq. (3) be Jacobi polynomials. Generally, Lévai [2] considered the differential equations

$$\frac{(g')^2}{(1-g^2)} = C, \quad \frac{(g')^2}{(1-g^2)^2} = C, \quad \frac{(g')^2 g}{(1-g^2)^2} = C \tag{6}$$

and used the Jacobi polynomials (eq. (22.6.1) of ref. [5]). Furthermore Lévai [2] classified the obtained potentials as PI, PII and PIII types. Then, Williams [3] followed this approach by using the third solution (eq. (22.6.3) of ref. [5]) to the hypergeometric equation, by considering the following differential equations:

$$\frac{(g')^2}{(1-g)^2} = C \quad \text{and} \quad \frac{(g')^2}{(1+g)^2} = C, \tag{7}$$

to find $g(r)$. More recently, we have presented a new class of analytical solvable potentials [4] by using the special cases of the Jacobi polynomials.

Here we shall consider the fourth solution (eq. (22.6.4) of ref. [5]) to eq. (3). So, we take the $F(g)$ function as

$$F(g) = \left(\sin\left(\frac{g}{2}\right)\right)^{\alpha+1/2}\left(\cos\left(\frac{g}{2}\right)\right)^{\beta+1/2} P_n^{\alpha,\beta}(\cos g), \tag{8}$$

where $P_n^{\alpha,\beta}(\cos g)$ is a Jacobi polynomial which satisfies (3) when

$$Q(g(r)) = 0 \tag{9}$$

and

$$R(g) = \frac{1-4\alpha^2}{16\sin^2(\frac{g}{2})} + \frac{1-4\beta^2}{16\cos^2(\frac{g}{2})} + \left(n + \frac{\alpha+\beta+1}{2}\right)^2. \tag{10}$$

Substituting (9) and (10) into (5), we obtain

$$E - V(r) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \frac{1-4\alpha^2}{16}\frac{(g')^2}{\sin^2(\frac{g}{2})} + \frac{1-4\beta^2}{16}\frac{(g')^2}{\cos^2(\frac{g}{2})}$$
$$+ \left(n + \frac{\alpha+\beta+1)}{2}\right)^2 (g')^2. \tag{11}$$

In order to identity the potential and energy, we can use different kinds of $g(r)$ functions. Here, we shall consider the case

$$\frac{(g')^2}{\cos^2(\frac{g}{2})} = C\,, \tag{12}$$

where C is a constants.

The solution to (12) is given by

$$g(r) = 2\arccos\left[\frac{2b\exp(-ar)}{1+b^2\exp(-2ar)}\right] \tag{13}$$

where a and b are arbitrary constants. By substituting into (11) we obtain E and $V(r)$ and the corresponding wave functions

$$V(r) = \frac{a^2b^2\left(1-16\left(n+\frac{\alpha+\beta+1}{2}\right)^2\right)\exp(-2ar)}{(1+b^2\exp(-2ar))^2} + \frac{a^2b^2(4\alpha^2-1)\exp(-2ar)}{(1-b^2\exp(-2ar))^2} \tag{14}$$

and

$$E = -a^2\beta^2\,, \tag{15}$$

where α and β are Jacobi polynomials parameters which have the initial restrictions $\alpha, \beta > -1$. As can be seen from (14) $V(r)$ is dependent on the quantum number n. We can remove this dependence as follow: For a given fixed α we redefine $a^2b^2(4\alpha^2-1)$ as a constant B. Furthermore, we set

$$-\beta = \alpha + 1 + 2n - \frac{\sqrt{1+\frac{A}{a^2b^2}}}{2}\,. \tag{16}$$

Where A is another constant. By substitution, we have the new type four-parameter potential, independent of n,

$$V(r) = -\frac{A\exp(-2ar)}{(1+b^2\exp(-2ar))^2} + \frac{B\exp(-2ar)}{(1-b^2\exp(-2ar))^2}\,, \tag{17}$$

where a and b are defined above, and, A and B are given as

$$A = a^2b^2\left(16\left(n+\frac{\alpha+\beta+1}{2}\right)^2 - 1\right) \tag{18}$$

and

$$B = a^2b^2(4\alpha^2-1)\,. \tag{19}$$

It should be noted that for given positive A, B, (or $\alpha > 1/2, \beta > -1$) and any a, b there is an extra condition on the quantum number n,

$$-\alpha+\frac{\sqrt{1+\frac{A}{a^2b^2}}}{2}>2n\,, \tag{20}$$

in order that the potential (17) have a number of bound states. Combining these parameters (eqs. (18) and (19)) and by substitution into (15), we obtain, the energy spectra as

$$E_n=-a^2\left[2n+1+\frac{1}{2}\left(-\sqrt{1+\frac{A}{a^2b^2}}+\sqrt{1+\frac{B}{a^2b^2}}\right)\right]^2. \tag{21}$$

From (2), (4), (8) and (13) once can find that the corresponding unnormalised wave functions that vanish at $r=\infty$ are

$$\Psi(r)\simeq\frac{1}{\sqrt{2a}}(u)^{\alpha}(1-u^2)^{(\beta+1)}P_n^{\alpha,\beta}(2u^2-1)\,. \tag{22}$$

Here

$$u=\frac{2b\exp(-ar)}{1+b^2\exp(-2ar)}\,.$$

The quadratic integrability of the wave functions (22) depends on their behavior as $r\to 0$. The Jacobi functions in (22) are well-behaved, for all values of $\alpha>0$ and $\beta>-1$, as $r\to 0$.

Finally, it is interesting to compare the Pöschl–Teller potential [6–9] with the four-parameter new potential in eq. (17). By substituting $i\lambda$ for a and 1 for b^2, one can rewrite the potential (17) as

$$V(r)=-\frac{A/4}{\cos^2(\lambda r)}-\frac{B/4}{\sin^2(\lambda r)}\,. \tag{23}$$

This form of the P–T potential has negative energies and it is a special case of the potential (17). However, under these conditions the potential parameters A and B (eqs. (18) and (19)), and the parameters of the P–T potential, change their signs. We note that by choosing appropriate parameters, the attractive part of our potential (17) is reduced to the attractive part of the Tietz potential [10], and also to the Rosen–Morse potential [11]. Moreover, one can find other slightly different analytically solvable exponential-type potentials in the literature [12,13].

For applications, the four constants appearing in the potential (17) can be determined by the following three conditions for two atomic molecules:

$$V'(r_e)=0,\quad V(\infty)-V(r_e)=D_e,\quad V''(r_e)=k_e\,, \tag{24}$$

and also from the rotational-vibrational coupling constants

$$\alpha_e=-[Xr_e/3+1]\left(\frac{6B_e^2}{\omega_e}\right)=F\left(\frac{6B_e^2}{\omega_e}\right), \tag{25}$$

where $X = V'''(r_e)/V''(r_e), \omega_e$ is the vibrational frequency and B_e the rotational constant. If we express the four above mentioned constants by means of the Sutherland parameter, $\Delta = k_e r_e^2/2D_e$ and the quantity $\Gamma = \frac{1}{9}X^2 r_e^2$, then we obtain for the four above mentioned constants

$$y_e^2 = \frac{\pm\sqrt{\Gamma/\Delta} - 1}{1 \pm \sqrt{\Gamma/\Delta}}, \quad a = \pm\frac{\sqrt{\Delta}}{2r_e}, \quad b^2 = y_e e^{\pm\sqrt{\Delta}},$$

$$B = \frac{D_e b^2}{4}\frac{(1-y_e)^4}{y_e^2} \quad \text{and} \quad A = \left(\frac{1+y_e}{1-y_e}\right)^4 B. \tag{26}$$

We also have for the anharmonicity

$$\omega_e X_e = 8\Delta \frac{W}{r_e^2 \mu_A}. \tag{27}$$

Although the value of α_e – we use experimental data to find the fourth parameter, – is different from the results when the Morse and the P–T potential are used [13], the value of $\omega_e X_e$ turns out to be the same.

Our result is therefore the construction of a new type of analytically solvable potential which is called the generalized P–T potential.

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